$3U_n = 2nU_{n-1} - 2nU_n \Rightarrow (3+2n)U_n = 2nU_{n-1}$.

4°/ Montrons par récurrence que $\forall n \in \mathbf{N}$, $U_n = \dfrac{2^{2n+2}\, n!(n+1)!}{(2n+3)!}$

• pour $n=0$, $U_0 = \dfrac{2}{3} = \dfrac{2^{2\cdot 0+2} 0!\,(0+1)!}{(2\cdot 0+3)!}$

donc la relation demandée est vraie pour $n=0$.

• supposons que pour un entier donné $n$ on a : $U_n = \dfrac{2^{2n+2}\, n!(n+1)!}{(2n+3)!}$

et montrons que $U_{n+1} = \dfrac{2^{2n+4}(n+1)!(n+2)!}{(2n+5)!}$

On a : $U_{n+1} = \dfrac{(2n+2)}{2n+5} U_n$ d'après 3°/ b)

$$U_{n+1} = \frac{(2n+2)}{2n+5} \cdot \frac{2^{2n+2}\, n!(n+1)!}{(2n+3)!} = \frac{2(n+1)(2n+4)2^{2n+2}\, n!(n+1)!}{(2n+5)!}$$

$$= \frac{2^2 (n+1)(n+2)2^{2n+2}\, n!(n+1)!}{(2n+5)!} = \frac{2^{2n+4}(n+1)!(n+2)!}{(2n+5)!}$$

d'où $\forall n \in \mathbf{N}$, $U_n = \dfrac{2^{2n+2}\, n!(n+2)!}{(2n+3)!}$.

**Solution 4 :**

1°/ $U_0 = \int_0^{\frac{\pi}{2}} \cos^0 t\, dt = \int_0^{\frac{\pi}{2}} 1\, dt = \dfrac{\pi}{2}$

$U_1 = \int_0^{\frac{\pi}{2}} \cos t\, dt = [\sin t]_0^{\frac{\pi}{2}} = 1$

$U_2 = \int_0^{\frac{\pi}{2}} \cos^2 t\, dt = \int_0^{\frac{\pi}{2}} \dfrac{1-\cos 2t}{2}\, dt = \left[\dfrac{t}{2} - \dfrac{1}{4}\sin 2t\right]_0^{\frac{\pi}{2}} = \dfrac{\pi}{4}$.

2°/ $U_{n+2} = \int_0^{\frac{\pi}{2}} \cos^{n+2} t\, dt$

Soient $\begin{cases} u' = \cos t \\ v = \cos^{n+1} t \end{cases}$ et $\begin{cases} u = \sin t \\ v' = -(n+1)\cos^n t \sin t \end{cases}$